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AN OPERATOR ANALYSIS OF A SUPERSYMMETRIC EFFECTIVE THEORY

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Abstract

An analysis of a $SU(2)_L \times SU(2)_R$ invariant, supersymmetric effective theory is given. The resulting leading and next to leading independent invariants are stated in terms of the underlying Killing vectors and Kähler potential. The appendices are devoted to the relationship between this geometrical point of view and the standard unitary matrix formulation.

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Effective Lagrangian techniques have been increasingly employed to describe lower energy behavior of models assuming there are strongly interacting dynamics underlying the theory. In this way, deviations from low energy behavior can be parametrized. A prominent example of this occurs in the electroweak chiral Lagrangian. In this case, if the electroweak symmetry breaking occurs because of a heavy Higgs bosonic sector, a low energy expansion can be made in powers of momentum.

This type of expansion has been studied in detail with invariants calculated to $O(p^4)$[1,2]. In this case, since the model respects a $SU(2)_L \times U(1)$ symmetry, the fields can be grouped into a dimensionless two by two unitary matrix and its complex conjugate. These matrices have properties that allow a reasonably simple determination of the invariants and indeed, two $O(p^2)$ and fourteen $O(p^4)$ terms have been enumerated for the gauged symmetry. For the case of a global $SU(2)_L \times SU(2)_R$ symmetry, the number of invariants drops to one $O(p^2)$ and two $O(p^4)$ terms.

However in supersymmetric models, the fields become complex chiral superfields. This presents a complication in the scheme presented by Longhitano[1] to categorize independent invariants since the convenient matrix properties exhibited in the electroweak case no longer hold. For the case of $SU(2)_L \times SU(2)_R$ global symmetry broken to the vector subgroup $SU(2)_V$, one can proceed analogous to Longhitano by defining the unitary matrices

$$\begin{aligned} U(\vec{x},\theta,\bar{\theta}) &= f exp(i\vec{T}\cdot\vec{\xi}) \\ U^{-1}(\vec{x},\theta,\bar{\theta}) &= f exp(-i\vec{T}\cdot\vec{\xi}) \end{aligned} \qquad (1)$$

where $\vec{\xi}$ is the multiplet of chiral superfields, $\vec{T} = \frac{1}{2}\vec{\sigma}$, and $\vec{\sigma}$ are the Pauli matrices. The superfield multiplet obeys the constraint $\det(U) = f^2$ where f^2 is the nonzero vacuum expectation value of the broken generator. And, an

invariant group element is then $V = U\bar{U}$ or $V^{-1} = \bar{U}^{-1}U^{-1}$. This constitutes the "standard" choice of coordinates ξ.

One can in principle evaluate all of the possible $SU(2)_L \times SU(2)_R$ terms eliminating the dependent invariants by use of integration by parts, equations of motion, and general matrix relations. And, this is tenable for the leading corrections in the action. They are

$$
\begin{aligned}
&Tr(D^\alpha V D_\alpha V) \\
&Tr(D^\alpha V^{-1} D_\alpha V)\,.
\end{aligned}
\tag{2}
$$

Then the general action to leading order can be stated as

$$
\begin{aligned}
S(\vec{x}) \;=\; & \int d^4x d^2\theta^2 d^2\bar{\theta}^2 \, a_m (TrV)^m [1 + b_1 Tr(D^\alpha V D_\alpha V) \\
& + b_2 Tr(D^\alpha V^{-1} D_\alpha V) + ...]\,,
\end{aligned}
\tag{3}
$$

which restates previous results[3] up to linear combinations and total partial derivatives.

Because of the supersymmetric vectorial measure, dimensional analysis on the component fields reveals that these invariants have dimension three. This means that in this scheme the bosonic component fields A carry dimension zero, the fermionic fields ψ carry dimension $1/2$, and the auxiliary bosonic fields F carry dimension 1. This counting procedure is equivalent to assigning a dimension of $1/2$ to every supersymmetric derivative in the action while assigning dimension 0 to each chiral superfield. So, unlike the bosonic case where the possible dimensions can only be even, the supersymmetric case allows for odd dimension terms. This implies that in the ordinary bosonic limit where all fermionic fields are taken to be zero and the scalar fields are reduced to Goldstone fields only, the odd dimension terms must vanish.

When this scheme is employed to find the subleading terms in this derivative expansion, the number of possible terms becomes so large as to make the calculation intractable. The problem arises not out of enumerating the terms which is straightforward, but in establishing the independent set of terms. This is in general a difficult question to answer.

Another formulation of the problem is to couch the model in geometrical terms. If the purely non-supersymmetric bosonic model describes a global symmetry G broken down to an invariant subgroup H, then a symmetric space can be formed with the Goldstone fields corresponding to the broken generators acting as the coordinates on the coset manifold G/H[4]. Then the usual Killing vectors, metric, Riemann and Ricci tensors, etc. can be defined on this manifold. Using this framework, Longhitano's two $SU(2)_L \times SU(2)_R$ invariant $O(p^4)$ terms can be restated as

$$\begin{aligned} & g_{ij} g_{kl} \partial^\mu \phi^i \partial_\mu \phi^j \partial^\nu \phi^k \partial_\nu \phi^l \\ & g_{ik} g_{jl} \partial^\mu \phi^i \partial_\mu \phi^j \partial^\nu \phi^k \partial_\nu \phi^l \end{aligned} \tag{4}$$

where g_{ij} is the metric of this manifold.

Extending this formulation to supersymmetry, it has been shown that the manifold formed by the now complex Goldstone fields is in fact Kählerian[5,6]. This means that there exists a real function $K(\phi, \bar{\phi})$ called the Kähler potential such that

$$g_{i\bar{j}} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} \, . \tag{5}$$

The form of the Kähler potential is restricted by the symmetry group G whose action on the coordinates can be specified by Killing vectors where $\delta_A \phi^i = A^i_A$. The choice of Killing vectors is not unique and in fact there are an infinite set of vectors to choose from. However, different choices of

Killing vectors only correspond to different nonlinear realizations of the global symmetry group which are physically equivalent. So, it is convenient to choose the set of Killing vectors to be

$$
\begin{aligned}
A^i_A &= \delta^a_A \epsilon_{aij} \phi^j + \delta^j_A (\delta^i_j + \phi^i \phi^j) \\
\bar{A}^{\bar{i}}_A &= \delta^a_A \epsilon_{a\bar{i}\bar{j}} \bar{\phi}^{\bar{j}} + \delta^{\bar{j}}_A (\delta^{\bar{i}}_{\bar{j}} + \bar{\phi}^{\bar{i}} \bar{\phi}^{\bar{j}})
\end{aligned}
\tag{6}
$$

with the capital indices running over the full group G while lower case indices toward the beginning of the alphabet count over the invariant subgroup H. Lower case indices toward the middle of the alphabet run over the remaining broken generators or the broken coset G/H. This choice in turn restricts the form of the Kähler potential to be[8]

$$
K(\phi, \bar{\phi}) = K' \left(\frac{1 + \phi \cdot \bar{\phi}}{\sqrt{1+\phi^2}\sqrt{1+\bar{\phi}^2}} \right) + F(\phi^2) + \bar{F}(\bar{\phi}^2)\,. \tag{7}
$$

Here, F and K' are arbitrary functions of their arguments with K' obeying the extra condition that $g_{i\bar{j}}(0,0) = \delta_{i\bar{j}}$. Note also that this choice of Killing vectors does not correspond to the standard coordinate choice of Longhitano's paper and will be denoted "geometrical" coordinates for lack of a better name. The connection between the two coordinate systems will be elaborated on later in this paper.

To construct invariants then, the complete set of independent tensors must be constructed and then contracted in every independent way. To this end, we make an expansion in powers of momentum which, as discussed before, translates to an expansion in number of covariant derivatives acting on the fields. For $O(p^3)$ terms, a basis set of tensors is formed from $D^\alpha \phi^i D_\alpha \phi^j$ and its complex conjugate. This contracted with all possible independent tensorial functions, $T_{ij}(\phi, \bar{\phi})$, results in the complete set of invariants to

this order. To be more specific, we also have to consider the possibility of having a basis tensor of the form $\mathcal{D}^2\phi^i$ so that the general tensor can be expressed as $T_i\mathcal{D}^2\phi^i$. Here, $\mathcal{D}^2$ is the covariant derivative defined to be $\mathcal{D}^2\phi^i = D^2\phi^i + \omega^i_{jk}D^\alpha\phi^j D_\alpha\phi^k$ where ω^i_{jk} is the Chiral connection defined in appendix A. But sinc! e this falls under the supersymmetric vectorial measure, this invariant can be transformed into $T_{i;j}D^\alpha\phi^i D_\alpha\phi^j$ via integration by parts. In the notation used by this paper, $T_{i;j}$ denotes one covariant derivative with respect to the field ϕ^j of the tensor T_i.

The tensorial objects $T_{ij}(\phi,\bar{\phi})$ can be formed from all possible tensors, direct products, covariant coordinate derivatives, and contractions among tensors. In a Kähler manifold with complex coordinates, there are a number of candidates that must be considered including the Riemann tensor $R^i_{j\bar{k}l}(\phi,\bar{\phi})$, Kähler metric $g_{i\bar{j}}(\phi,\bar{\phi})$, the so called "Chiral" metric $\gamma_{ij}(\phi)$, the "Chiral" Riemann tensor $W_{ijkl}(\phi)$, and the Ricci tensor $R_{ij}(\phi,\bar{\phi})$ [2]. Another candidate that must be considered is chosen from the arbitrary set of functions $K'(\phi,\bar{\phi})$ where the function is required to be G-invariant. So we define the scalar function,

$$K_0(\phi,\bar{\phi}) = \ln(1+\bar{\phi}\phi) - \frac{1}{2}\ln(1+\phi^2) - \frac{1}{2}\ln(1+\bar{\phi}^2) \tag{8}$$

on which n covariant derivatives form an n-rank tensor. Notice that there are an infinite number of choices of scalar quantities that can be defined for this role since $K'(\phi,\bar{\phi})$ is an arbitrary function. However, covariant derivatives of any of these choices can be related up to scalar factors to $K_0(\phi,\bar{\phi})$. These scalar factors can then be absorbed in the action at zero order. This procedure amounts to a field redefinition in the action.

[2] For a discussion of these tensors see reference [9].

In addition, since every tensor transforming under this group can be expressed in terms of δ_{ij},$\phi^i\phi^j$, and ϵ_{ijk}, one would expect a contribution to the set of tensors antisymmetric with respect to its indices. Such objects can be constructed from the chiral and antichiral vielbeins of the space which can be expressed as

$$\begin{aligned} e_{\underline{m}}^{\underline{l}} &= \frac{1}{1+\phi^2}(\delta^{lm}+\epsilon_{lmn}\phi^n) \\ \bar{e}_{\bar{m}}^{\underline{l}} &= \frac{1}{1+\bar{\phi}^2}(\delta^{lm}+\epsilon_{lmn}\bar{\phi}^n)\,. \end{aligned} \tag{9}$$

The underlined indices refer to a tangent space index which transforms linearly while the remaining index transforms nonlinearly under the broken symmetries. Given this, we define the tensor

$$T_{ijk} = \epsilon_{mnp} e_{i}^{\underline{m}} e_{j}^{\underline{n}} e_{k}^{\underline{p}} \tag{10}$$

along with its complex conjugate which has a component that is proportional to ϵ_{ijk} and so totally antisymmetric under interchange of i,j, or k.

From this list of tensors, it can be proven that the independent set of tensors up to direct products are

$$\begin{aligned} & K_{0;i} \\ \gamma_{ij} &\equiv A_{Ai}A_{Aj} \\ g_{i\bar{j}} &\equiv K_{0;i\bar{j}} \\ T_{ijk} &\equiv \epsilon_{mnp} e_{i}^{\underline{m}} e_{j}^{\underline{n}} e_{k}^{\underline{p}} \\ S_{ij} &\equiv T_{ijk}K_0^{;k} \\ A_{ij\bar{k}} &\equiv T_{ijm}\gamma^{mn}g_{n\bar{k}} \\ I_{j\bar{k}} &\equiv A_{ij\bar{k}}K_0^{;i} \end{aligned} \tag{11}$$

along with complex conjugates where necessary [3]. No further contractions or covariant derivatives can be taken that generate an independent tensor. Therefore, the possible tensors at any order can be easily specified as either one of the these tensors or a direct product of a combination of them. As a result, the $O(p^3)$ terms can be immediately written down as

$$
\begin{aligned}
&K_{0;i}K_{0;j}D^{\alpha}\phi^i D_{\alpha}\phi^j \\
&\gamma_{ij}D^{\alpha}\phi^i D_{\alpha}\phi^j
\end{aligned}
\tag{12}
$$

along with their complex conjugates. Note that it is assumed that each invariant is associated with a complex coefficient so the action will remain hermitian. This list reproduces the result of equation (2) in geometric form.

It is worth noting at this point the connection between the geometrical and the matrix (standard coordinate) notations. These two cases really represent just a change in non-linear realization of the global symmetry group. Thus, given a suitable redefinition of the Goldstone fields ϕ^i, a translation can be made[8]. In this case if we define the scalar invariant

$$a(\phi,\bar{\phi}) = \frac{1+\bar{\phi}\phi}{\sqrt{1+\phi^2}\sqrt{1+\bar{\phi}^2}}\,, \tag{13}$$

then

$$Tr(V) = 2a(\phi,\bar{\phi})\,. \tag{14}$$

Furthermore, defining ξ^i to be a field in standard coordinates and ϕ^i to be a field in the geometrical notation of this paper, translations of any tensorial quantity can be made with the assignment $\xi^i = \frac{2}{\phi}\arctan(\phi)\phi^i$. Note that $\phi = \sqrt{\phi\cdot\phi}$.

[3]For a more complete description of these tensors, see appendix A.

Now, we can proceed to the subleading $O(p^4)$ terms. This case adds a few complications in that there are four covariant derivatives acting on the fields as well as the four derivatives from the vector measure which allows for more base tensors as well as more complicated tensors $T_{ijkl}(\phi,\bar\phi)$. However, the tensors $T_{ijkl}(\phi,\bar\phi)$ are still formed by direct products of the original independent tensors of equation (11). In addition, it must be realized that the derivatives acting on the fields are covariant derivatives which require extra care. For example, the Grassmannian property $D^3\phi = 0$ does not hold for covariant derivatives $\mathcal{D}^3\phi \neq 0$. Another consideration is the fact that the bases at this order are not independent due to integration by parts. Nevertheless, the evaluation of terms are still tractable in the geometrical notation and in fact there are 42 terms:

$$
\begin{array}{ll}
1.\ K_{0;i}K_{0;j}\mathcal{D}^2\phi^i\mathcal{D}^2\phi^j & 14.\ K_{0;i}I_{j\bar k}\bar D^{\dot\alpha}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k} \\
2.\ \gamma_{ij}\mathcal{D}^2\phi^i\mathcal{D}^2\phi^j & 15.\ K_{0;i}g_{j\bar k}\bar D^{\dot\alpha}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k} \\
3.\ K_{0;i}K_{0;\bar j}\mathcal{D}^2\phi^i\bar{\mathcal{D}}^2\bar\phi^{\bar j} & 16.\ K_{0;i}K_{0;j}K_{0;\bar k}K_{0;\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
4.\ g_{i\bar j}\mathcal{D}^2\phi^i\bar{\mathcal{D}}^2\bar\phi^{\bar j} & 17.\ g_{i\bar k}g_{j\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
5.\ I_{i\bar j}\mathcal{D}^2\phi^i\bar{\mathcal{D}}^2\bar\phi^{\bar j} & 18.\ K_{0;i}K_{0;j}\bar\gamma_{\bar k\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
6.\ K_{0;i}\gamma_{jk}\mathcal{D}^2\phi^i D^\alpha\phi^j D_\alpha\phi^k & 19.\ K_{0;i}K_{0;\bar k}g_{j\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
7.\ K_{0;i}K_{0;j}K_{0;k}\mathcal{D}^2\phi^i D^\alpha\phi^j D_\alpha\phi^k & 20.\ \gamma_{ij}\bar\gamma_{\bar k\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
8.\ \gamma_{ij}K_{0;\bar k}D^\alpha\phi^i D_\alpha\phi^j\bar{\mathcal{D}}^2\bar\phi^{\bar k} & 21.\ K_{0;i}K_{0;\bar k}I_{j\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
9.\ K_{0;i}g_{j\bar k}D^\alpha\phi^i D_\alpha\phi^j\bar{\mathcal{D}}^2\bar\phi^{\bar k} & 22.\ I_{i\bar k}I_{j\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
10.\ K_{0;i}K_{0;j}K_{0;\bar k}D^\alpha\phi^i D_\alpha\phi^j\bar{\mathcal{D}}^2\bar\phi^{\bar k} & 23.\ I_{i\bar k}g_{j\bar l}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k}\bar D^{\dot\alpha}\bar\phi^{\bar l} \\
11.\ K_{0;i}I_{j\bar k}D^\alpha\phi^i D_\alpha\phi^j\bar{\mathcal{D}}^2\bar\phi^{\bar k} & 24.\ K_{0;i}K_{0;j}\gamma_{kl}D^\alpha\phi^i D_\alpha\phi^j D^\beta\phi^k D_\beta\phi^l \\
12.\ A_{ij\bar k}\bar D^{\dot\alpha}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k} & 25.\ \gamma_{ij}\gamma_{kl}D^\alpha\phi^i D_\alpha\phi^j D^\beta\phi^k D_\beta\phi^l \\
13.\ S_{ij}K_{0;\bar k}\bar D^{\dot\alpha}D^\alpha\phi^i D_\alpha\phi^j\bar D_{\dot\alpha}\bar\phi^{\bar k} & 26.\ S_{ik}S_{jl}D^\alpha\phi^i D_\alpha\phi^j D^\beta\phi^k D_\beta\phi^l
\end{array}
\tag{15}
$$

The other terms come from complex conjugates where necessary. As a check, we require this list to recover Longhitano's result in the bosonic limit and it can be shown that we do indeed recover the invariants of equation (4) when this constraint is implemented. Also in this limit, the $O(p^3)$ terms vanish as expected.

It should be noted that nothing has been said about the use of field equations to reduce the number of independent operators of the problem on shell. This has been done because no assumption has been made as to the order in momentum to which this expansion is to be truncated which in turn establishes an order to which the field equations are valid. It should be stressed that terms eliminated with the use of a lower order field equation will not necessarily be eliminated in a higher order calculation. With this in mind, we choose the expansion to truncate at $O(p^4)$ which establishes the field equations as

$$\begin{aligned} \mathcal{D}^2\phi^i &= 2K_{0;j}D^\alpha\phi^i D_\alpha\phi^j + O(p^4) \\ \bar{\mathcal{D}}^2\bar{\phi}^{\bar{i}} &= 2K_{0;\bar{j}}\bar{D}_{\dot\alpha}\bar{\phi}^{\bar{i}}\bar{D}^{\dot\alpha}\bar{\phi}^{\bar{j}} + O(p^4)\,. \end{aligned} \tag{16}$$

This condition acts to eliminate all of the $O(p^4)$ terms containing second derivatives. Furthermore, it and its complex conjugate establish a relationship between the $O(p^3)$ and $O(p^4)$ terms that removes two more operators on shell. Thus, we choose to eliminate the terms

$$\begin{gathered} K_{0;i}K_{0;j}\bar{\gamma}_{\bar{k}\bar{l}}D^\alpha\phi^i D_\alpha\phi^j\bar{D}_{\dot\alpha}\bar{\phi}^{\bar{k}}\bar{D}^{\dot\alpha}\bar{\phi}^{\bar{l}} \\ \gamma_{ij}K_{0;\bar{k}}K_{0;\bar{l}}D^\alpha\phi^i D_\alpha\phi^j\bar{D}_{\dot\alpha}\bar{\phi}^{\bar{k}}\bar{D}^{\dot\alpha}\bar{\phi}^{\bar{l}}\,. \end{gathered} \tag{17}$$

It must be noted as well that these equations constitute a constraint on the superfields only under the vectorial measure. Since the remaining terms in

the action still fall under the vectorial measure, there are components that can still be eliminated with the field equation. However since this would destroy the explicit supersymmetric construction of the Lagrangian, we refrain from applying this condition here. So the field equation further reduces the list to a final result of 21 invariants.

$$
\begin{array}{ll}
1. & K_{0;i}g_{j\bar{k}}\bar{D}^{\dot{\alpha}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}} \\
2. & A_{ij\bar{k}}\bar{D}^{\dot{\alpha}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}} \\
3. & S_{ij}K_{0;\bar{k}}\bar{D}^{\dot{\alpha}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}} \\
4. & K_{0;i}I_{j\bar{k}}\bar{D}^{\dot{\alpha}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}} \\
5. & K_{0;\bar{i}}g_{\bar{j}k}D^{\alpha}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{j}}D_{\alpha}\phi^k \\
6. & \bar{A}_{\bar{i}\bar{j}k}D^{\alpha}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{j}}D_{\alpha}\phi^k \\
7. & \bar{S}_{\bar{i}\bar{j}}K_{0;k}D^{\alpha}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{j}}D_{\alpha}\phi^k \\
8. & K_{0;\bar{i}}I_{\bar{j}k}D^{\alpha}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{j}}D_{\alpha}\phi^k \\
9. & K_{0;i}K_{0;j}K_{0;\bar{k}}K_{0;\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
10. & g_{i\bar{k}}g_{j\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
11. & K_{0;i}K_{0;\bar{k}}g_{j\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
12. & \gamma_{ij}\bar{\gamma}_{\bar{k}\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
13. & K_{0;i}K_{0;\bar{k}}I_{j\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
14. & I_{i\bar{k}}I_{j\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
15. & I_{i\bar{k}}g_{j\bar{l}}D^{\alpha}\phi^i D_{\alpha}\phi^j \bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{l}} \\
16. & K_{0;i}K_{0;j}\gamma_{kl}D^{\alpha}\phi^i D_{\alpha}\phi^j D^{\beta}\phi^k D_{\beta}\phi^l \\
17. & \gamma_{ij}\gamma_{kl}D^{\alpha}\phi^i D_{\alpha}\phi^j D^{\beta}\phi^k D_{\beta}\phi^l \\
18. & S_{ik}S_{jl}D^{\alpha}\phi^i D_{\alpha}\phi^j D^{\beta}\phi^k D_{\beta}\phi^l \\
19. & K_{0;\bar{i}}K_{0;\bar{j}}\bar{\gamma}_{\bar{k}\bar{l}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}}\bar{D}_{\dot{\beta}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\beta}}\bar{\phi}^{\bar{l}} \\
20. & \bar{\gamma}_{\bar{i}\bar{j}}\bar{\gamma}_{\bar{k}\bar{l}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}}\bar{D}_{\dot{\beta}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\beta}}\bar{\phi}^{\bar{l}} \\
21. & \bar{S}_{\bar{i}\bar{k}}\bar{S}_{\bar{j}\bar{l}}\bar{D}_{\dot{\alpha}}\bar{\phi}^{\bar{i}}\bar{D}^{\dot{\alpha}}\bar{\phi}^{\bar{j}}\bar{D}_{\dot{\beta}}\bar{\phi}^{\bar{k}}\bar{D}^{\dot{\beta}}\bar{\phi}^{\bar{l}}
\end{array}
\tag{18}
$$

This list is considerably longer than the 10 $O(p^4)$ terms of reference [3][4]. It represents a comprehensive list of the number of independent operators to this order which are unexpectedly large in comparison with the Electroweak case.

As a next step, one could constrain the model to obey a $SU(2)_L \times U(1)$ symmetry retaining terms to the same $O(p^4)$ order. Such a model would be an extension of the model presented by reference [10] and lies within current experimental constraints on the extensions of the Standard Model. Since the invariants listed in this paper also obey this new symmetry, one would expect a more complex Lagrangian as a result. This exercise will be left for a future paper.

[4]See appendix B for a discussion of this.

Appendix A:
Tensor construction in the geometrical basis

The key to successfully constructing a set of invariants relies on constructing the most general set of tensors such that only direct products give rise to new tensors. In this way, much of the work of determining independence of possible candidates can be partitioned into a question of tensorial independence and connections among invariants by integration by parts. The easiest way to approach the problem of tensorial independence is to work in the simplest coordinate system one can find so that the dependences are not overly complicated. The geometrical coordinates used in this paper result in the simplest forms of the tensors and were chosen precisely for that reason.

We start with the simplest tensorial quantity that can be defined for the case of $SU(2)_L \times SU(2)_R$ invariance which is the scalar. As seen by equation (7), there is only one combination of the fields ϕ and $\bar{\phi}$ that results in a scalar invariant aside from arbitrary powers of the invariant. So we define the scalar

$$a \equiv \frac{\sqrt{\beta\bar{\beta}}}{\alpha} \tag{A.1}$$

where

$$\begin{aligned} \beta &= \frac{1}{1+\phi^2} \\ \bar{\beta} &= \frac{1}{1+\bar{\phi}^2} \\ \alpha &= \frac{1}{1+\phi\cdot\bar{\phi}}\,. \end{aligned} \tag{A.2}$$

So the function K_0 can be simply stated as $K_0 = \ln(a)$. Thus, one derivative (or one covariant derivative) creates a rank 1 tensor:

$$K_{0;i} = \alpha\bar{\phi}^i - \beta\phi^i\,. \tag{A.3}$$

Applying another derivative with respect to the complex conjugated field results in the metric of the space:

$$g_{\bar{i}j} \equiv K_{0;\bar{i}j} = \alpha(\delta^{\bar{i}j} - \alpha\phi^{\bar{i}}\bar{\phi}^j)\,. \tag{A.4}$$

Notice that this metric is not formed from the contraction of two Killing vectors. Thus, we must define another two index tensor which we shall denote the Chiral metric:

$$\gamma_{ij} \equiv A_{Ai}A_{Aj} = \beta(\delta^{ij} - \beta\phi^i\phi^j)\,. \tag{A.5}$$

Taking covariant derivatives of a tensorial quantity creates a higher rank tensor. Unfortunately, the covariant derivative is not uniquely defined beyond one derivative and in fact there are two ways to construct a covariant derivative. The first employs the usual connection defined for the space which is defined from the Kähler potential.

$$\Gamma^i_{jk} \equiv g^{i\bar{n}}K_{0,\bar{n}jk} = -\alpha(\bar{\phi}^k\delta^i_j + \bar{\phi}^j\delta^i_k)\,. \tag{A.6}$$

But another connection can be defined which shall be called the Chiral connection[9]. It is defined by

$$\omega^i_{jk} \equiv \frac{1}{2}\gamma^{in}(\gamma_{nk,j} + \gamma_{nj,k} - \gamma_{jk,n}) = -\beta(\phi^k\delta^i_j + \phi^j\delta^i_k)\,. \tag{A.7}$$

In these equations, a comma denotes an ordinary derivative with respect to the field. Both of these objects deserve the apellation connection since they transform as Christoffel symbols. Thus, two covariant derivatives can be defined. For example, a covariant derivative of a first rank tensor could be

$$T_{i;j} \equiv T_{i,j} - \omega^m_{ij}T_m \tag{A.8}$$

or

$$T_{i;j'} \equiv T_{i,j} - \Gamma^m_{ij} T_m \,. \tag{A.9}$$

However, it can be shown that $T_{i;j'} = T_{i;j} + K_{0;i} T_j + K_{0,j} T_i$ and so the two definitions are related. This procedure can be implemented to all orders of covariant derivatives which allows us the freedom to choose one definition. In this paper, we chose to use covariant differentiation defined by the Chiral connection.

If we apply covariant derivatives to the given tensors defined so far, we find that this set of tensors closes up to direct products. For example, we have

$$\begin{aligned} \gamma_{ij;k} &= 0 \\ K_{0;ij} &= -K_{0;i} K_{0,j} - \gamma_{ij} \\ g_{\bar{i}j;k} &= -g_{\bar{i}j} K_{0;k} - g_{\bar{i}k} K_{0;j} \,. \end{aligned} \tag{A.10}$$

We have not addressed the issue of contractions among these tensors. However, it is seen that these tensors do in fact contract up in such a way as to produce another object in the list or direct products thereof. This indicates we have a "basis" set of tensors that are independent and can be used as building blocks of more complex tensors via direct products.

However, this is not all of the possible tensor expressions. If one considers a general function that transforms as a tensor under the group $SU(2)_L \times SU(2)_R$, one finds that it can be expressed as combinations of ϕ^i,δ^{ij}, and ϵ_{ijk}. This arises from the fact that traces of Pauli matrices lead to functions of this sort. So one might expect a tensorial quantity arising from a ϵ_{ijk} component. And indeed, such tensors have been expressed in the main body of this paper all of which arise from the tensor T_{ijk}. There, we defined

it in terms of the vielbeins of the space, however for the purposes of the appendices, it would be simpler to extract out the essential quantity of T_{ijk}; the component proportional to ϵ_{ijk}. In this spirit, we redefine T_{ijk} to be

$$T_{ijk} = \beta^2 \epsilon_{ijk} \,. \tag{A.11}$$

The other antisymmetric tensors arise naturally from contractions of T_{ijk} with our previously defined set of tensors and they have the form:

$$\begin{aligned} S_{ij} &= \alpha\beta\epsilon_{ijk}(\bar{\phi}^k - \phi^k) \\ A_{ij\bar{k}} &= \frac{1}{a^2}\beta^2\bar{\beta}(\epsilon_{ij\bar{k}} + \epsilon_{ijm}\phi^k\phi^m + \epsilon_{jkm}\phi^i\bar{\phi}^m - \epsilon_{ikm}\phi^j\bar{\phi}^m) \\ I_{i\bar{j}} &= \beta\bar{\beta}\epsilon_{i\bar{j}k}(\bar{\phi}^k - \phi^k) \end{aligned} \tag{A.12}$$

As a whole, these seven tensors constitute a closed set; closed in the sense that no contractions or covariant derivatives result in a new tensor. Therefore, these seven tensors constitute the necessary building blocks to form invariants under this group.

As a final confirmation of the completeness of this set, we can evaluate some basic tensors of the manifold. The Riemann tensor can be constructed as[11]

$$R_{\bar{i}j\bar{k}l} \equiv g_{\bar{i}n}\Gamma^n_{jl,\bar{k}} = -[g_{\bar{i}j}g_{\bar{k}l} + g_{\bar{i}l}g_{\bar{k}j}] \,. \tag{A.13}$$

Similarly, the Chiral Riemann tensor has the form

$$\begin{aligned} W_{ijkl} &\equiv \gamma_{im}[\omega^m_{jl,k} - \omega^m_{jk,l} + \omega^m_{nk}\omega^n_{jl} - \omega^m_{nl}\omega^n_{jk}] \\ &= \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk} \,. \end{aligned} \tag{A.14}$$

And finally, the Ricci tensor for a general Kähler manifold can be derived from the metric of the space[12]:

$$\begin{aligned} R_{i\bar{j}} &\equiv \frac{\partial^2}{\partial\phi^i\partial\bar{\phi}^{\bar{j}}} \ln\det(g_{\bar{k}l}) \\ &= -4g_{i\bar{j}} \,. \end{aligned} \tag{A.15}$$

Appendix B:
A translation of coordinates

Although the standard coordinates are typically written in terms of the Pauli matrices, this by no means disallows a reformulation of the coordinates into a geometrical way of looking at them. The unitary matrix $exp(i\vec{T}\cdot\vec{\xi})$ can be written as $U = cos(\frac{\xi}{2}) + \frac{2i}{\xi}sin(\frac{\xi}{2})\vec{\xi}\cdot\vec{T}$. So if we were to proceed as in Appendix A, we would take

$$Tr(U\bar{U}) = 2\left[cos(\frac{\xi}{2})cos(\frac{\bar{\xi}}{2}) + sin(\frac{\xi}{2})sin(\frac{\bar{\xi}}{2})\frac{\bar{\xi}\cdot\xi}{\bar{\xi}\xi}\right] \tag{B.1}$$

and define that as our scalar invariant, take covariant derivatives, and proceed to build a tensor list as before. This gives rise to a completely equivalent "geometrical" interpretation of the action. For example, the Chiral metric defined in appendix A has the form in standard coordinates:

$$\begin{aligned}\gamma_{ij}(\xi) &= \frac{\partial\xi^i}{\partial\phi^m}\frac{\partial\xi^j}{\partial\phi^n}\gamma^{mn}(\phi) \\ &= \delta^{ij}\frac{t^2}{(1+t^2)\xi^2} - \xi^i\xi^j\left(\frac{t^2}{(1+t^2)\xi^4} - \frac{1}{4\xi^2}\right)\end{aligned} \tag{B.2}$$

where $t = \tan(\frac{\xi}{2})$. In general, if we parametrized the Killing vectors of a field ϕ to be

$$A^i_A = \delta^a_A\epsilon_{aij}\phi^j + \delta^j_A\left[\delta^i_j g(\phi) + \phi^i\phi^j h(\phi)\right] \tag{B.3}$$

then a general Chiral metric can be defined as

$$\gamma^{ij} = \delta^{ij}(g^2+\phi^2) + \phi^i\phi^j(h^2\phi^2 + 2gh - 1)\,. \tag{B.4}$$

An examination of this equation reveals that the choice of $f = 1$ and $g = 1$ gives the metric in its simplest form which is also the geometrical choice of coordinates used in this paper.

We can use this kind of redefinition to formulate a translation from the standard coordinates to the geometric coordinates. We have already seen that $Tr(V) = 2a$ is a relation between the two field definitions accomplished via the field redefinition $\xi^i = \frac{2}{\phi}\arctan(\phi)\phi^i$. So, using this translation directly on U and $\bar{U}$ leads to the result:

$$\begin{aligned} U &= \sqrt{\beta}(1 + 2i\vec{\phi}\cdot\vec{T}) \\ \bar{U} &= \sqrt{\bar{\beta}}(1 - 2i\vec{\bar{\phi}}\cdot\vec{T})\,. \end{aligned} \tag{B.5}$$

So in principle, one can take any standard coordinate invariant and translate it into the geometrical coordinate equivalent by taking the appropriate derivatives and traces. One could, for example, take the list generated by reference [3] and find the geometrical coordinate equivalent. And indeed, a couple of the simplest terms are:

$$\begin{aligned} \bar{D}^2 Tr\bar{Z}\,Tr\bar{Z}\,Tr\bar{Z} &= -\frac{1}{8}\bar{D}^2 D^2 a^3 (K_{0;i}\mathcal{D}^2\phi^i - \gamma_{ij}D^\alpha\phi^i D_\alpha\phi^j) \\ &\quad \times(K_{0;k}\mathcal{D}^2\phi^k - \gamma_{kl}D^\alpha\phi^k D_\alpha\phi^l) \\ \bar{D}^2 D^2 TrZ\,Tr\bar{Z} &= \frac{1}{4}\bar{D}^2 D^2 a^2 (K_{0;i}\mathcal{D}^2\phi^i - \gamma_{ij}D^\alpha\phi^i D_\alpha\phi^j) \\ &\quad \times(K_{0;\bar{k}}\mathcal{D}^2\phi^{\bar{k}} - \bar{\gamma}_{\bar{k}\bar{l}}\bar{D}_{\dot{\alpha}}\phi^{\bar{k}}\bar{D}^{\dot{\alpha}}\phi^{\bar{l}}) \end{aligned} \tag{B.6}$$

where

$$\begin{aligned} Z &= -\frac{1}{4}\bar{D}^2 V \\ W_\alpha &= -\frac{1}{4}\bar{D}^2 D_\alpha V\,. \end{aligned} \tag{B.7}$$

As one can see, the two coordinates heavily mix upon translation making a one to one correspondence difficult. However, one can count the invariants generated by the list, and with a little effort, one finds that the 10 $O(p^4)$

terms[5] involve 23 of the invariants listed in equation (15).

One might be interested to see if any of the other invariants not part of this list can be generated in the standard coordinates and upon consideration it can be shown that this is the case. There are various combinations of matrices that can be traced to immediately see a tensorial component. A short and incomplete list is:

$$
\begin{aligned}
Tr(D^\alpha V) &= 2aK_{0;i} D^\alpha \phi^i \\
Tr(D^\alpha V D_\alpha V) &= \left[4a^2 K_{0;i} K_{0;j} - 2\gamma_{ij}\right] D^\alpha \phi^i D_\alpha \phi^j \\
Tr(V D^\alpha V D^\beta V) &= -2aS_{ij}\, D^\alpha \phi^i D^\beta \phi^j + ... \\
Tr(\bar{D}^{\dot{\alpha}} D^\alpha V) &= 2a\left[K_{0;i} K_{0;\bar{j}} + g_{i\bar{j}}\right] \bar{D}^{\dot{\alpha}} \bar{\phi}^{\bar{j}} D^\alpha \phi^i + 2aK_{0;i} \bar{D}^{\dot{\alpha}} D^\alpha \phi^i \\
Tr(V \bar{D}^{\dot{\alpha}} D^\alpha V) &= 2I_{i\bar{j}}\, \bar{D}^{\dot{\alpha}} \bar{\phi}^{\bar{j}} D^\alpha \phi^i + ... \qquad \text{(B.8)}
\end{aligned}
$$

From this, it becomes evident that many of the invariants listed in equation (15) can be immediately generated including, for example, a couple that are not created from combinations of the matrices Z and W_α:

$$
\begin{aligned}
Tr(V D^\alpha V D^\beta V)\, Tr(V D_\alpha V D_\beta V) &= 4aS_{ik} S_{jl} D^\alpha \phi^i D_\alpha \phi^j D^\beta \phi^k D_\beta \phi^l + ... \\
Tr(V \bar{D}^{\dot{\alpha}} D^\alpha V)\, Tr(\bar{D}_{\dot{\alpha}} D_\alpha V) &= -4aK_{0;i} I_{j\bar{k}}\, \bar{D}^{\dot{\alpha}} D^\alpha \phi^i D_\alpha \phi^j \bar{D}_{\dot{\alpha}} \bar{\phi}^{\bar{k}} + ... \qquad \text{(B.9)}
\end{aligned}
$$

So it can be concluded that all independent invariants of this group cannot be written solely as functions of Z and W_α.

[5] Actually only 8 of the 10 terms are independent since two of them can be eliminated with the use of $\det M = \frac{1}{2}[Tr(M)Tr(M) - Tr(M^2)]$ where M is a general 2×2 matrix.

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